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Renormalized canonical perturbation theory for stochastic propagators

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Abstract. A canonical transformation which removes the coherent oscillatory motion of a particle in a stochastic potential (the renormalized oscillation-centre transformation) is constructed by a new classical perturbation method using Lie operators and Green function techniques. A frequency and wavevector dependent particle–wave collision operator is calculated explicitly for stationary, homogeneous electrostatic turbulence in the short wavelength limit. The width of the resonance is proportional to the one-third power of the quasilinear diffusion coefficient, in agreement with Dupree's 1966 result. However the k dependence is quite different from that expected from a simple Wiener process model. In fact, at large k spatial diffusion dominates over velocity diffusion, in sharp contrast with previous theories.

1. Introduction

It has been remarked by several authors (Cook and Sanderson 1974, Rolland 1976) that the weak coupling theory of Dupree (1966) and Weinstock (1969) of stochastic acceleration of a particle in a random potential (as in a plasma) is valid only in the same limit as unrenormalized quasilinear theory (i.e. in the case of a broad spectrum of weak waves). Various authors have improved on the original theory, but the basic picture of the average particle motion being a Wiener process (i.e. $\langle(\Delta p)^2\rangle \propto t$, $\langle(\Delta x)^2\rangle \propto t^3$) remains unchanged, except that Rolland (1976) suggests that it breaks down when partial trapping of the particles in troughs of the waves occurs. The reason for this is intuitively clear since trapping induces vortex motion in phase space, thus producing behaviour more akin to two-dimensional Brownian motion in phase space ($\langle(\Delta x)^2\rangle \propto t$, $\langle(\Delta p)^2\rangle \propto t$).

A logical way to proceed beyond the weak turbulence approximation is to develop a general perturbation theory for the single particle propagator since this contains all the information needed to calculate response functions, correlation functions, etc. Such a formalism has been set up by Marcuvitz (1974), and independently by Cook (1975). Secularities are removed by adding an effective collision term to both sides of the equation of motion for the propagator, thus introducing resonance broadening into the Green function and eliminating secularities. We term this method *additive renormalization*. The Marcuvitz–Cook expansion does not appear to have been much exploited, and we can discern two reasons for this: (i) their Green function obeys a non-Markovian equation, and (ii) spatial diffusion does not occur at any finite order of the expansion. In the present paper we show that (i) is not an insuperable problem, by finding an

asymptotic solution in the large k limit, while (ii) may be overcome by applying a canonical transformation to the system before solving for the Green function. We term this *multiplicative renormalization.*

The transformed Hamiltonian is a rapidly varying function of momentum, thus introducing spatial diffusion at lowest order. The resulting theory of resonance broadening gives a broadening similar to that of Dupree in the broad spectrum limit, but the physical mechanism is very different. We find that, at short wavelength, spatial diffusion is always the dominant broadening mechanism, while at medium wavelengths it is competitive with velocity space diffusion. That is, the Wiener process model is *never* valid.

The canonical transformation employed is similar to the 'oscillation-centre' transformation (Dewar 1973, Johnston 1976), but differs from it because an operator method (Deprit 1969) is used rather than conventional generating-function methods, and, more importantly, because it is combined with the Marcuvitz–Cook renormalization scheme to give a self-consistent and causal generalized Hamilton–Jacobi equation, rather than the *ad hoc* acausal filter-function method used in the oscillation-centre theory. To distinguish the present transformation from the previous one we term it the *renormalized oscillation-centre transformation.*

Kawakami (1970), Kawakami and Yagashita (1971), and Kawakami and Sanuki (1971) have used an operator technique due to Hori (1966) to achieve what is in effect an oscillation-centre transformation. This appears to be essentially the same as our unrenormalized theory, with a slightly different filter function. There is no resonance broadening (although there is a *shift* in the position of the resonance) because a renormalized Green function has not been used.

The basic idea involved is in some ways similar to that behind the 'interaction picture' frequently used to take out some unwanted part of the motion by a unitary factorization of the propagator. However the part we take out is the *coherent oscillatory motion*, whereas the interaction picture factorizes out the unperturbed motion. Since all the coherent motion is removed, the oscillation-centre motion is the truly stochastic part of the orbit—there is no need for phase-mixing arguments to justify omission of non-resonant terms. Also, because most of the interaction is removed, the mean square interaction Hamiltonian is smaller after the transformation, and convergence of the series for the new Green function should be faster than that for the untransformed one. Put another way, our Green function is much closer to the exact oscillation-centre propagator than is the untransformed Green function because the untransformed Green function 'smears out' from its initial delta function character very rapidly, owing to the coherent response to non-resonant waves.

In § 2 we set up an operator method for canonical transformations, and in § 3 we specify how the generating function is to be constructed and the new Hamiltonian obtained. In § 4 we review the renormalized perturbation theory for the Green function of a particle moving in a stochastic external potential. No attempt is made to obtain self-consistency with Maxwell's equations. Such a postponement of self-consistency implies a self-consistent field or Vlasov approximation to the system, since polarization effects are thereby ignored.

In § 5 we derive 'quasilinear' equations for the single particle Green function and particle–wave collision operator for a particle moving in a stationary, homogeneous spectrum of waves. The solution at large k is found in § 6, where we also make a rough estimate for medium k. We find the width of the resonance to scale as the one-third power of the quasilinear diffusion coefficient, as predicted by Dupree (1966), but we

find that the asymptotic behaviour at large k is that the width increases as k, rather than $k^{2/3}$ as predicted by Dupree's analysis.

Taking this behaviour as evidence of partial trapping leads to a seeming paradox since, in the broad spectrum limit, the correlation time of the non-resonant waves (as seen by the particle) is much less than the 'trapping time'. Thus one might expect the particle to be scattered out of a potential trough before it has time to bounce. We believe the resolution is because much of the response to the non-resonant waves averages out—the average momentum is to some extent an adiabatic invariant. The oscillation-centre transformation brings out this invariance as much as possible.

2. Canonical transformation operator

Following Deprit (1969) we define the *Lie derivative* L_W generated by a function $W(q, p, t)$ by its action on an arbitrary function $\phi(q, p, t)$

$$L_W\phi = \{\phi, W\}, \tag{1a}$$

where $\{\phi, W\}$ is the Poisson bracket of ϕ and W. That is, in a phase space of dimension $2N$,

$$L_W = \sum_{i=1}^{N} \left(\frac{\partial W}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial W}{\partial q_i}\frac{\partial}{\partial p_i}\right). \tag{1b}$$

(Normally, of course, $N = 3$, but the results of this section actually apply to any Hamiltonian system of arbitrary dimensionality.)

We also define the adjoint operator $L_W^\dagger$ by

$$L_W^\dagger = \sum_{i=1}^{N} \left(\frac{\overleftarrow{\partial}}{\partial q_i}\frac{\partial W}{\partial p_i} - \frac{\overleftarrow{\partial}}{\partial p_i}\frac{\partial W}{\partial q_i}\right) \tag{2}$$

where the arrows indicate that the derivatives act to the left. If we consider L_W and $L_W^\dagger$ to be operators in the Hilbert space of functions $\phi(q, p, t)$, then L_W is an anti-Hermitian operator (Prigogine 1962)

$$L_W^\dagger = -L_W. \tag{3}$$

Suppose that W also depends on some parameter ϵ (such as the charge on the particle) so that

$$W \equiv W(q, p, t; \epsilon). \tag{4}$$

Then we define the *canonical transformation operator* C_W by the operator equation

$$\partial C_W/\partial\epsilon = L_W C_W, \tag{5a}$$

with the initial condition

$$C_W(\epsilon = 0) = 1. \tag{5b}$$

In terms of C_W the solution to the inhomogeneous equation

$$\partial A/\partial\epsilon = L_W A + B \tag{6}$$

is

$$A = C_W \int_0^\epsilon C_W^{-1} B \, \mathrm{d}\epsilon + C_W A(\epsilon = 0), \tag{7}$$

where A and B can be either functions or operators, and C_W^{-1} is the *inverse canonical transformation operator*, defined by the equation

$$\partial C_W^{-1}/\partial\epsilon = -C_W^{-1}L_W, \tag{8a}$$

and the initial condition

$$C_W^{-1}(\epsilon = 0) = 1. \tag{8b}$$

It is a simple matter to verify that

$$C_W C_W^{-1} = C_W^{-1} C_W = 1, \tag{9}$$

by differentiation with respect to ϵ. If we take the adjoint of equations (5), and use equation (3), we find that $C_W^\dagger$ obeys the defining equations (8) for C_W^{-1}. Hence C_W is a unitary operator

$$C_W^\dagger = C_W^{-1}. \tag{10}$$

It is of interest to compare our operator C_W with the operator E_W of Deprit (1969), defined by

$$E_W\phi = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!}\left[\left(\frac{\partial}{\partial\epsilon} + L_W\right)^n \phi\right]_{\epsilon=0}. \tag{11}$$

By differentiating with respect to ϵ it is easy to show that

$$\partial E_W/\partial\epsilon = E_W L_W, \tag{12}$$

whence we have, by equations (1) and (8)

$$E_W = C_{-W}^{-1}. \tag{13}$$

We find that the perturbation theory is simpler in terms of C_W, and also C_W has a closer analogy with the time evolution operator of quantum mechanics. Thus we can write down the solution of equation (5a) by analogy (see, e.g., Bjorken and Drell 1965):

$$\begin{aligned} C_W &= 1 + \int_0^\epsilon d\epsilon_1\, L_W(\epsilon_1) + \int_0^\epsilon d\epsilon_1 \int_0^{\epsilon_1} d\epsilon_2\, L_W(\epsilon_1)L_W(\epsilon_2) + \ldots \\ &= \mathscr{E} \exp\left(\int_0^\epsilon L_W\, d\epsilon\right), \end{aligned} \tag{14a}$$

where $\mathscr{E}$ is the ϵ *ordering operator.* The inverse operator is constructed trivially from equation (14a) by unitarity

$$C_W^{-1} = 1 - \int_0^\epsilon d\epsilon_1\, L_W(\epsilon_1) + \int_0^\epsilon d\epsilon_1 \int_0^{\epsilon_1} d\epsilon_2\, L_W(\epsilon_2)L_W(\epsilon_1) - \ldots. \tag{14b}$$

To show that C_W does in fact generate a canonical transformation, define the transformed variables q_i^* and p_i^* by

$$q_i^*(q, p, t) = C_W q_i, \qquad p_i^*(q, p, t) = C_W p_i. \tag{15}$$

If we can show that the fundamental Poisson bracket relations are preserved then the transformation is canonical. That this is the case follows from the theorem

$$C_W\{\phi, \psi\} = \{C_W\phi, C_W\psi\} \tag{16}$$

for arbitrary ϕ and ψ. Equation (16) is readily proved by showing that the difference of

the right- and left-hand sides obeys equation (6) with $B = 0$ and $A(\epsilon = 0) = 0$, and thus vanishes for all ϵ by equation (7). Use has been made of Jacobi's identity in the form

$$L_W\{\phi, \psi\} = \{L_W\phi, \psi\} + \{\phi, L_W\psi\}. \tag{17}$$

It is similarly possible to show that

$$C_W\phi(q, p) = \phi(C_W q, C_W p) \tag{18}$$

for arbitrary ϕ. Using equations (1) and (9) we can rewrite equation (16) as the operator identity

$$C_W L_\phi C_W^{-1} = L_{C_W\phi}. \tag{19a}$$

By replacing ϕ with $C_W^{-1}\phi$ we also find

$$C_W^{-1} L_\phi C_W = L_{C_W^{-1}\phi}. \tag{19b}$$

Sudarshan and Mukunda (1974) show that equations (16) and (18) are consequences of the fact that the canonical transformations form a realization of a Lie group.

In the next section we shall have occasion to use the identity

$$\partial C_W/\partial t = L_V C_W, \tag{20}$$

where

$$V \equiv C_W \int_0^\epsilon C_W^{-1}(\partial W/\partial t)\, \mathrm{d}\epsilon. \tag{21}$$

By differentiating equation (4) with respect to t and by using equations (6) and (7) we can show

$$\partial C_W/\partial t = C_W \int_0^\epsilon C_W^{-1} L_{\partial W/\partial t} C_W \, \mathrm{d}\epsilon,$$

which can be manipulated into the form of equation (20) by use of equations (19*a*) and (19*b*).

3. Renormalized oscillation-centre transformation

In equation (15) we take q_i^* and p_i^* to be the *exact* phase-space coordinates of a particle with Hamiltonian $H(q^*, p^*, t)$, consisting of an average part H_0 and a fluctuating part $\tilde{H}$. The coordinates q_i, p_i are to be the phase-space coordinates of the *oscillation centre* (Dewar 1973, Johnston 1976) which moves like a particle with Hamiltonian $K(q, p, t)$, which also has an average part $\bar{K}$ and a fluctuating part $\tilde{K}$. Physically the oscillation centre is the averaged position of the particle, with the coherent response to $\tilde{H}$ removed. Thus $\tilde{K}$ is to be a purely stochastic driving term. It is the purpose of this section to construct a canonical transformation conforming to this physical picture.

Since the oscillation centre is to be always close to the exact position we require W to be always a small quantity. This is partly ensured by requiring that W average to zero:

$$\langle W(1)\rangle = 0, \tag{22}$$

where the argument 1 is an abbreviation for the 'world point' (q_{1i}, p_{1i}, t_1), $i = 1, 2, \ldots, N$ (and similarly for 2, 3, etc.). The angle brackets $\langle\ \rangle$ denote the ensemble average.

As C_W maps the oscillation-centre coordinates into the exact coordinates, it is actually the inverse of the oscillation-centre transformation. We shall call this the *clothing transformation*, and C_W the *clothing operator*, since we may think of C_W as clothing (or dressing) the bare particle with its fluctuation cloud. This is very similar to the 'dressed test particle' picture (Hubbard 1961, Rostoker 1964a, b).

As an alternative to thinking of particle coordinates we can work with the exact single-particle distribution function $f^*(q, p, t)$ and the oscillation-centre distribution function $f(q, p, t)$. Because the transformation is canonical, the distribution function is an invariant, $f^*(q^*, p^*, t) = f(q, p, t)$, or, by equations (10) and (18)

$$f^*(q, p, t) = C_W^{-1} f(q, p, t). \tag{23}$$

(Note that the operator mapping f into f^* is the inverse of the operator mapping q and p into q^* and p^*.) The two distribution functions obey the Liouville equations

$$\partial f^*/\partial t + L_H f^* = 0, \tag{24}$$

$$\partial f/\partial t + L_K f = 0. \tag{25}$$

Multiplying equation (23) by C_W and substituting for f in equation (25) we find

$$(L_V C_W - C_W L_H + L_K C_W) f^* = 0, \tag{26}$$

where V is given by equation (21), and we have used equation (24). Using equation (19*a*) we see that equation (26) can be satisfied for arbitrary f^* if and only if

$$K = C_W H - V \tag{27}$$

(to within a constant which we have set to zero).

By multiplying equation (27) by C_W^{-1} and differentiating with respect to ϵ we find an equation for W

$$\frac{\partial W}{\partial t} + L_K W = C_W \frac{\partial H}{\partial \epsilon} - \frac{\partial K}{\partial \epsilon}. \tag{28}$$

This is simply a different version of equation (27), and does not define W in any way, unless K is specified somehow. We could require that K be independent of the q_i, making the p_i constants of the motion, but this would make W secular due to resonances. We avoid this by allowing K to have the fluctuating part $\tilde{K}$, which we take to be a *linear functional of* W. That is

$$K = \bar{K} + \tilde{K}, \tag{29}$$

where

$$\tilde{K}(1) = \int_0^\epsilon \mathrm{d}\epsilon \int \mathrm{d}2\, \Sigma(1|2) W(2), \qquad \mathrm{d}2 \equiv \mathrm{d}^N q_2\, \mathrm{d}^N p_2\, \mathrm{d}t_2. \tag{30}$$

Substitution of equation (30) in equation (28) gives

$$\left(\frac{\partial}{\partial t_1} + L_{\bar{K}(1)}\right) W(1) + \int \mathrm{d}2\, \Sigma(1|2)\, W(2) = S(1), \tag{31}$$

with the source term S given by

$$S(1) \equiv C_{W(1)} \frac{\partial H(1)}{\partial \epsilon} + L_{W(1)} \tilde{K}(1) - \frac{\partial \bar{K}(1)}{\partial \epsilon}. \tag{32}$$

It is now seen that $\Sigma(1|2)$ must be chosen to be a generalized 'collision' kernel representating the scattering of the particle by resonant waves. Outside the resonant region $\Sigma(1|2)$ is zero, $\tilde{K}(1)$ vanishes, and Deprit's (1969) perturbation theory applies. A way of constructing $\Sigma(1|2)$ will be given in the next section where we also define a Green function, $G(1|2)$, for the operator acting on W on the left-hand side of equation (31). We are free to choose initial conditions so that W is *defined* by the particular solution

$$W(1) = \int \mathrm{d}2\; G(1|2)S(2). \tag{33}$$

Define the two-point function $F(1|2)$ by the equation

$$F(1|2) \equiv \int \mathrm{d}3\; \Sigma(1|3)G(3|2). \tag{34}$$

Then, from equations (30) and (33):

$$\tilde{K}(1) = \int_0^{\epsilon} \mathrm{d}\epsilon \int \mathrm{d}2\; F(1|2)S(2). \tag{35}$$

It is seen from equation (35) that $F(1|2)$ replaces the filter function $\Phi(t_1 - t_2)$ introduced *ad hoc* in unrenormalized oscillation-centre theory (Dewar 1973). In the present theory $F(1|2)$ is determined self-consistently. An even more important change is that $F(1|2)$ is a *causal* function, unlike $\Phi(t_1 - t_2)$ which, being an even function of $t_1 - t_2$, was acausal.

In order to satisfy equation (22), with $W(1)$ given by equation (33), we must require that

$$\frac{\partial \bar{K}(1)}{\partial \epsilon} = \left\langle C_{W(1)} \frac{\partial H(1)}{\partial \epsilon} + L_{W(1)} \tilde{K}(1) \right\rangle. \tag{36}$$

Equations (32), (33), (35) and (36) can be solved to arbitrary order in ϵ by power series expansion, thus completing the specification of the transformation.

We shall give the first few terms in the series. Following Deprit (1969) we write

$$W = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} W_{n+1}, \qquad S = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} S_{n+1}. \tag{37}$$

Defining

$$L_1 = L_{W_1}, \qquad L_2 = L_{W_2}, \qquad \text{etc}, \tag{38}$$

equations (14*a*) and (14*b*) give

$$\begin{aligned} C_W &= 1 + \epsilon L_1 + \tfrac{1}{2}\epsilon^2 (L_1^2 + L_2) + \tfrac{1}{6}\epsilon^3 (L_1^3 + L_1 L_2 + 2L_2 L_1 + L_3) + \ldots \\ C_W^{-1} &= 1 - \epsilon L_1 + \tfrac{1}{2}\epsilon^2 (L_1^2 - L_2) - \tfrac{1}{6}\epsilon^3 (L_1^3 - L_2 L_1 - 2L_1 L_2 + L_3) + \ldots . \end{aligned} \tag{39}$$

We also write

$$H = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} H_n. \tag{40}$$

When $\tilde{K}$ is calculated from equation (35) it will be a series of the form

$$\tilde{K} = \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} K_n \tag{41}$$

while (36) will give

$$\bar{K} = H_0 + \sum_{n=1}^{\infty} \frac{\epsilon^{2n}}{(2n)!} \bar{K}_{2n}. \tag{42}$$

We treat $G(1|2)$ and $F(1|2)$ as O(1) quantities and use operator notation for brevity (e.g., equation (33) becomes $W = GS$), so that

$$\begin{aligned} S_1 &= H_1 \qquad S_2 = L_{GH_1} H_1' + H_2 - \bar{K}_2 \\ S_3 &= L_{GH_1}^2 H_1 + L_{GL_{GH_1}H_1} H_1'' + L_{GH_1} F L_{GH_1} H_1' + L_{G(H_2-\bar{K}_2)} H_1'' \\ &\quad + L_{GH_1} F(H_2 - \bar{K}_2) + 2L_{GH_1} H_2 + H_3 \end{aligned} \tag{43}$$

and so on, and

$$\bar{K}_2 = \langle L_{GH_1} H_1' \rangle, \tag{44}$$

where

$$H_1' \equiv H_1 + FH_1, \qquad H_1'' \equiv H_1 + 2FH_1. \tag{45}$$

We shall not give W and $\tilde{K}$ as they are trivially related to S by equations (33) and (35).

4. Single particle Green functions

We define the (retarded) propagator for the exact particle motion, $\hat{U}(1|2)$, by the equations

$$(\partial/\partial t_1)\hat{U}(1|2) + L_{H(1)}\hat{U}(1|2) = \delta(1|2), \tag{46}$$

$$\hat{U}(1|2) = 0, \qquad t_1 < t_2, \tag{47}$$

where $\delta(1|2) \equiv \delta^N(q_1 - q_2)\,\delta^N(p_1 - p_2)\,\delta(t_1 - t_2)$. Marcuvitz (1974) and Cook (1975) have shown how to set up a formal perturbation theory for $\hat{U}(1|2)$ using the average of $\hat{U}(1|2)$ as a Green function, but, since $\hat{U}(1|2)$ describes both coherent and stochastic motion, we believe the convergence to be slower than is possible if we instead solve for the *oscillation-centre propagator* $\hat{G}(1|2)$, defined by the equations

$$(\partial/\partial t_1)\hat{G}(1|2) + L_{K(1)}\hat{G}(1|2) = \delta(1|2), \tag{48}$$

$$\hat{G}(1|2) = 0, \qquad t_1 < t_2. \tag{49}$$

The two propagators are related by unitary transformation

$$\hat{G}(1|2) = C_{W(1)}\hat{U}(1|2)C^{\dagger}_{W(2)}. \tag{50}$$

Since equations (46) and (48) are identical, save for the replacement of H by K, the Marcuvitz–Cook perturbation theory can be taken over unchanged. The central idea is to introduce the Green function

$$G(1|2) = \langle \hat{G}(1|2) \rangle, \tag{51}$$

and to postulate that it obeys the Non-Markovian equation

$$\left(\frac{\partial}{\partial t_1} + L_{\bar{K}(1)}\right) G(1|2) + \int \mathrm{d}3\, \Sigma(1|3) G(3|2) = \delta(1|2), \tag{52}$$

where $\Sigma(1|3)$ is the effective collision kernel introduced in the previous section. Both

$G(1|2)$ and $\Sigma(1|2)$ are causal functions

$$G(1|2)=0, \qquad t_1<t_2; \qquad \Sigma(1|2)=0, \qquad t_1<t_2. \tag{53}$$

The reason for using $G(1|2)$ as a Green function is that it is the closest possible *non-stochastic* approximation to $\hat{G}(1|2)$.

As Rolland (1976) points out, $G(1|2)$ does not obey the group property, that is, for $t_1>\bar{t}>t_2$,

$$\int \mathrm{d}3\; G(1|3)\delta(t_3-\bar{t})G(3|2)\neq G(1|2), \tag{54}$$

and $G(1|2)$ should not, therefore, be called a propagator. However, $G(1|2)$ is still a Green function, and that is all we need. It follows directly from equation (52) that the solution to the equations

$$\begin{aligned}&(\partial\phi(1)/\partial t_1)+\int \mathrm{d}2\;\Sigma(1|2)\phi(2)=\sigma(1),\\ &\phi(1),\sigma(1)=0, \qquad t_1<t_0,\end{aligned} \tag{55}$$

is

$$\phi(1)=\int \mathrm{d}2\; G(1|2)\sigma(2). \tag{56}$$

We replace $\tilde{K}$ by $\lambda\tilde{K}$, so that

$$L_K=L_{\bar{K}}+\lambda L_{\tilde{K}} \tag{57}$$

where λ is a formal expansion parameter (to be set equal to 1 at the end), and expand Σ and G in the series

$$\Sigma=\lambda^2\Sigma^{(2)}+\lambda^3\Sigma^{(3)}+\ldots, \tag{58}$$

$$\hat{G}=G+\lambda G^{(1)}+\lambda^2 G^{(2)}+\ldots. \tag{59}$$

We substitute equations (58) and (59) into equation (48), equate equal powers of λ on the left- and right-hand sides, and use equations (52) and (55) to obtain

$$G^{(1)}=-GL_{\tilde{K}}G, \qquad G^{(2)}=G(L_{\tilde{K}}GL_{\tilde{K}}+\Sigma^{(2)})G, \tag{60}$$

and so on, where we have adopted the same operator notation as was used in equation (43). From equations (51) and (59) we need

$$\langle G^{(n)}(1|2)\rangle=0, \qquad n\geqslant 1. \tag{61}$$

To satisfy this for $n=2$ we need

$$\Sigma^{(2)}(1|2)=\langle L_{\tilde{K}(1)}G(1|2)L^{\dagger}_{\tilde{K}(2)}\rangle. \tag{62}$$

It is seen that $\Sigma^{(2)}$ is a generalized Fokker–Planck diffusion operator. Because all the coherent response (linear and non-linear) has already been taken out by the oscillation-centre transformation there is no need to continue the series further unless large-angle collisions are important. Since even trapping effects occur in a small region of phase space we conjecture that equation (62) will be adequate for most purposes. If we formally order ϵ to be smaller than any power of λ then equation (62) can be made the defining equation for $\Sigma(1|2)$ to all orders in ϵ, thus providing broadening for all resonances.

5. Stationary, homogeneous turbulence

As in quasilinear theory (Vedenov *et al* 1962, Drummond and Pines 1962) we work to first order in ϵ. We shall use the generalized Fokker–Planck collision operator of the previous section. From equations (35) and (43) we have

$$\tilde{K}(1) = \int \mathrm{d}2\, F(1|2) H_1(2). \tag{63}$$

The average part of K is given by equations (42) and (44), and can be written

$$\bar{K}(1) = H_0(1) + \Delta(1), \tag{64}$$

where

$$\Delta(1) = \frac{1}{2}\frac{\partial}{\partial \boldsymbol{x}_1} \cdot \int \mathrm{d}2 \left\langle (H_1(1) + \tilde{K}(1)) \frac{\partial}{\partial \boldsymbol{p}_1} G(1|2) H_1(2) \right\rangle$$
$$- \frac{1}{2}\frac{\partial}{\partial \boldsymbol{p}_1} \cdot \int \mathrm{d}2 \left\langle (H_1(1) + \tilde{K}(1)) \frac{\partial}{\partial \boldsymbol{x}_1} G(1|2) H_1(2) \right\rangle. \tag{65}$$

In a homogeneous system the first term vanishes.

We write equation (62) in conservation form

$$\Sigma(1|2) = \frac{\partial^2}{\partial \boldsymbol{x}_2 \, \partial \boldsymbol{x}_1} : \left(\left\langle \frac{\partial \tilde{K}(1)}{\partial \boldsymbol{p}_1} \frac{\partial \tilde{K}(2)}{\partial \boldsymbol{p}_2} \right\rangle G(1|2) \right) - \frac{\partial^2}{\partial \boldsymbol{x}_2 \, \partial \boldsymbol{p}_1} : \left(\left\langle \frac{\partial \tilde{K}(1)}{\partial \boldsymbol{x}_1} \frac{\partial \tilde{K}(2)}{\partial \boldsymbol{p}_2} \right\rangle G(1|2) \right)$$
$$- \frac{\partial^2}{\partial \boldsymbol{p}_2 \, \partial \boldsymbol{x}_1} : \left(\left\langle \frac{\partial \tilde{K}(1)}{\partial \boldsymbol{p}_1} \frac{\partial \tilde{K}(2)}{\partial \boldsymbol{x}_2} \right\rangle G(1|2) \right) + \frac{\partial^2}{\partial \boldsymbol{p}_2 \, \partial \boldsymbol{p}_1} : \left(\left\langle \frac{\partial \tilde{K}(1)}{\partial \boldsymbol{x}_1} \frac{\partial \tilde{K}(2)}{\partial \boldsymbol{x}_2} \right\rangle G(1|2) \right), \tag{66}$$

where we have taken the q_i to be Cartesian components of the position vector $\boldsymbol{x}$. We use a dyadic notation in which

$$\mathbf{A} : \mathbf{B} \equiv \sum_{i=1}^{N} \sum_{j=1}^{N} A_{ji} B_{ij}.$$

We assume the system to be homogeneous and stationary on average, so that all two point functions depend on $\boldsymbol{x}_1$, t_1 and $\boldsymbol{x}_2$, t_2 only through $\boldsymbol{x}_1 - \boldsymbol{x}_2$ and $t_1 - t_2$. Define the Fourier transform of an arbitrary function $A(\boldsymbol{x}_1 - \boldsymbol{x}_2, t_1 - t_2)$ by

$$A_{\boldsymbol{k},\omega} = \int\int \mathrm{d}^N x \, \mathrm{d}t \exp(-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x} + \mathrm{i}\omega t) A(\boldsymbol{x}, t),$$

so that

$$A(\boldsymbol{x}, t) = \int\int \frac{\mathrm{d}^N k \, \mathrm{d}\omega}{(2\pi)^{N+1}} \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x} - \mathrm{i}\omega t) A_{\boldsymbol{k},\omega}.$$

We make an exception in the case of $G_{\boldsymbol{k},\omega}$, which is defined by

$$\mathrm{i} G_{\boldsymbol{k},\omega}(\boldsymbol{p}_1, \boldsymbol{p}_2) = \int\int \mathrm{d}^N x \, \mathrm{d}t \exp(-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x} + \mathrm{i}\omega t) G(\boldsymbol{x}, \boldsymbol{p}_1, t|0, \boldsymbol{p}_2, 0), \tag{67}$$

the reason for the factor of i being to make $G_{\boldsymbol{k},\omega}$ real outside the resonant region.

Also needed are the Fourier transforms $I_{k,\omega}$ and $J_{k,\omega}$ of the interaction Hamiltonian correlation functions

$$I_{k,\omega}(p_1, p_2) \equiv \int\int \mathrm{d}^N x_1 \, \mathrm{d}t_1 \exp[-\mathrm{i}k \cdot (x_1 - x_2) + \mathrm{i}\omega(t_1 - t_2)] \langle H_1(1) H_1(2) \rangle \tag{68}$$

$$J_{k,\omega}(p_1, p_2) \equiv \int\int \mathrm{d}^N x_1 \, \mathrm{d}t_1 \exp[-\mathrm{i}k \cdot (x_1 - x_2) + \mathrm{i}\omega(t_1 - t_2)] \langle \tilde{K}(1) \tilde{K}(2) \rangle. \tag{69}$$

From equation (63) we see that these are related by the equation

$$J_{k,\omega}(p_1, p_2) = \int\int \mathrm{d}^N p_3 \, \mathrm{d}^N p_4 \, F_{k,\omega}(p_1, p_3) F^*_{k,\omega}(p_2, p_4) I_{k,\omega}(p_3, p_4), \tag{70}$$

where

$$F_{k,\omega}(p_1, p_2) = \mathrm{i} \int \mathrm{d}^N p_3 \, \Sigma_{k,\omega}(p_1, p_3) G_{k,\omega}(p_3, p_2). \tag{71}$$

After Fourier transformation, equations (52), (65) and (66) become

$$\left(\omega - k \cdot v_1 - k \cdot \frac{\partial \Delta}{\partial p_1}\right) G_{k,\omega}(p_1, p_2) + \mathrm{i} \int \mathrm{d}^N p_3 \, \Sigma_{k,\omega}(p_1, p_3) G_{k,\omega}(p_3, p_2) = \delta(p_1 - p_2) \tag{72}$$

$$\Delta(p_1) = \frac{1}{2} \frac{\partial}{\partial p_1} \cdot \int\int \frac{\mathrm{d}^N k \, \mathrm{d}\omega}{(2\pi)^{N+1}} \int \mathrm{d}^N p_2 \int \mathrm{d}^N p_3 \, k G_{k,\omega}(p_1, p_2) \times [\delta(p_1 - p_3) + F^*_{k,\omega}(p_1, p_3)] I_{k,\omega}(p_2, p_3), \tag{73}$$

$$\begin{aligned} \Sigma_{k,\omega}(p_1, p_2) = \mathrm{i} \int\int \frac{\mathrm{d}^N k' \, \mathrm{d}\omega'}{(2\pi)^{N+1}} & \left(G_{k-k',\omega-\omega'}(p_1, p_2) k \cdot \frac{\partial}{\partial p_1} k \cdot \frac{\partial}{\partial p_2} J_{k',\omega'}(p_1, p_2) \right) \\ & - \frac{\partial}{\partial p_1} \cdot \left(G_{k-k',\omega-\omega'}(p_1, p_2) k' k \cdot \frac{\partial}{\partial p_2} J_{k',\omega'}(p_1, p_2) \right) \\ & - \frac{\partial}{\partial p_2} \cdot \left(G_{k-k',\omega-\omega'}(p_1, p_2) k' k \cdot \frac{\partial}{\partial p_1} J_{k',\omega'}(p_1, p_2) \right) \\ & + \frac{\partial^2}{\partial p_1 \, \partial p_2} : [G_{k-k',\omega-\omega'}(p_1, p_2) k' k' J_{k',\omega'}(p_1, p_2)]. \end{aligned} \tag{74}$$

We have defined the unrenormalized velocity v_1 by

$$v_1 = \partial H_0(p_1)/\partial p_1. \tag{75}$$

Equations (70)–(74) form a coupled set of non-linear integral equations, whose solution determines the renormalized Green function and collision kernel self-consistently.

6. Asymptotic solution

In the present paper we shall be content to obtain a qualitative understanding of the solutions. To do this we work in the limit

$$k \gg k', \qquad |\omega| \gg |\omega'|, \tag{76}$$

where k' and ω' are a typical wavevector and frequency of a wave in the turbulent spectrum.

The first benefit of the limit (76) is that the last three terms of equation (74) can be neglected. Physically, this means that spatial diffusion is the dominant process at short wavelengths. The second advantage is that $G_{\boldsymbol{k}-\boldsymbol{k}',\omega-\omega'}$ can be replaced by $G_{\boldsymbol{k},\omega}$ and brought outside the integral. It is then consistent with both equation (72) and equation (74) to assume that $G_{\boldsymbol{k},\omega}$ and $\Sigma_{\boldsymbol{k},\omega}$ are proportional to $\delta(\boldsymbol{p}_1-\boldsymbol{p}_2)$

$$G_{\boldsymbol{k},\omega}(\boldsymbol{p}_1,\boldsymbol{p}_2)=G_{\boldsymbol{k},\omega}(\boldsymbol{p}_1)\delta(\boldsymbol{p}_1-\boldsymbol{p}_2), \tag{77}$$

and similarly for $\Sigma_{\boldsymbol{k},\omega}$. Henceforth $G_{\boldsymbol{k},\omega}$ and $\Sigma_{\boldsymbol{k},\omega}$ will be taken to denote $G_{\boldsymbol{k},\omega}(\boldsymbol{p})$ and $\Sigma_{\boldsymbol{k},\omega}(\boldsymbol{p})$, respectively.

Assuming $I_{\boldsymbol{k},\omega}$ to be independent of $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ (electrostatic approximation) we find that equations (72) and (74) reduce to

$$G_{\boldsymbol{k},\omega}=\frac{1}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}-\boldsymbol{k}\cdot\partial\Delta/\partial\boldsymbol{p}+\mathrm{i}\Sigma_{\boldsymbol{k},\omega}}, \tag{78}$$

and

$$\Sigma_{\boldsymbol{k},\omega}=\mathrm{i}\boldsymbol{k}\boldsymbol{k}:\mathscr{D}G_{\boldsymbol{k},\omega}, \tag{79}$$

where

$$\mathscr{D}\equiv\mathrm{Re}\int\!\!\int\frac{\mathrm{d}^Nk'\,\mathrm{d}\omega'}{(2\pi)^{N+1}}\frac{\partial F^*_{\boldsymbol{k}',\omega'}}{\partial\boldsymbol{p}}\frac{\partial F_{\boldsymbol{k}',\omega'}}{\partial\boldsymbol{p}}I_{\boldsymbol{k}',\omega'}. \tag{80}$$

We define $F_{\boldsymbol{k}',\omega'}$ by

$$F_{\boldsymbol{k}',\omega'}(\boldsymbol{p})\equiv\int\mathrm{d}^Np'\,F_{\boldsymbol{k}',\omega'}(\boldsymbol{p},\boldsymbol{p}'). \tag{81}$$

Solving equations (78) and (79) for $\Sigma_{\boldsymbol{k},\omega}$ we find

$$\begin{aligned}\mathrm{Re}\,\Sigma_{\boldsymbol{k},\omega}&=\tfrac{1}{2}\theta(4\boldsymbol{k}\boldsymbol{k}:\mathscr{D}-\Omega_{\boldsymbol{k}}^2)(4\boldsymbol{k}\boldsymbol{k}:\mathscr{D}-\Omega_{\boldsymbol{k}}^2)^{1/2}\\ \mathrm{Im}\,\Sigma_{\boldsymbol{k},\omega}&=\tfrac{1}{2}\Omega_{\boldsymbol{k}}-\tfrac{1}{2}\theta(\Omega_{\boldsymbol{k}}^2-4\boldsymbol{k}\boldsymbol{k}:\mathscr{D})(\Omega_{\boldsymbol{k}}^2-4\boldsymbol{k}\boldsymbol{k}:\mathscr{D})^{1/2}\end{aligned} \tag{82}$$

where $\theta(x)$ is the Heaviside step function, and

$$\Omega_{\boldsymbol{k}}\equiv\omega-\boldsymbol{k}\cdot\boldsymbol{v}-\boldsymbol{k}\cdot(\partial\Delta/\partial\boldsymbol{p}).$$

Equation (82) shows that the real part of $\Sigma_{\boldsymbol{k},\omega}$ (which gives the effective damping due to particle–wave collisions) vanishes outside the range

$$2(\boldsymbol{k}\boldsymbol{k}:\mathscr{D})^{1/2}<|\Omega_k|<2(\boldsymbol{k}\boldsymbol{k}:\mathscr{D})^{1/2}.$$

Thus the width of the resonance goes as k in the short wavelength limit. As already noted, spatial diffusion is dominant in this limit, so it is not surprising that the width of the resonance should scale differently from the $k^{2/3}$ behaviour expected from a simple Wiener process model. Note however that the width does not scale according to a Brownian motion model with constant spatial diffusion coefficient either. This model would predict k^2 behaviour.

Strictly speaking we cannot calculate $\mathscr{D}$ from this asymptotic analysis because $F_{\boldsymbol{k}',\omega'}$ cannot be calculated without violating the condition (76). Nevertheless we can make a

rough estimate. First note from equation (82) that at large $|\Omega_k|$ we can approximate $\Sigma_{k,\omega}$ by

$$\Sigma_{k,\omega} \sim \mathrm{i}\boldsymbol{k}\boldsymbol{k} : \mathscr{D}/\Omega_k.$$

When k is comparable with k' we cannot replace $\Sigma_{k-k',\omega-\omega'}$ with $\Sigma_{k,\omega}$ in equation (74). The result of the convolution integral will be to smooth out the discontinuous behaviour of $\Sigma_{k,\omega}$, so that it will be approximately given by

$$\Sigma_{k',\omega'} \simeq \frac{\mathrm{i}\boldsymbol{k}'\boldsymbol{k}' : \mathscr{D}}{\Omega_{k'} + 2\mathrm{i}\delta'}. \tag{83}$$

We assume that the width is, to within a constant of order unity, given by $(\boldsymbol{k}'\boldsymbol{k} : \mathscr{D})^{1/2}$, so we can write

$$\delta' = (\boldsymbol{k}'\boldsymbol{k}' : \mathscr{D})^{1/2} \cos\theta', \tag{84}$$

where θ' is a constant of order unity.

Provided that the turbulent spectrum has a sufficiently wide range of phase velocities (much greater than the trapping width), it is valid to retain equation (77) as an approximation. In this case it can be shown also that $\boldsymbol{k}' \cdot \partial\Delta/\partial\boldsymbol{p}$ is negligible in comparison with δ'. It then follows from equations (71) and (78) that

$$F_{k',\omega'} = \frac{-\boldsymbol{k}'\boldsymbol{k}' : \mathscr{D}}{(\omega' - \boldsymbol{k}' \cdot \boldsymbol{v} + \mathrm{i}\delta_+)(\omega' - \boldsymbol{k}' \cdot \boldsymbol{v} + \mathrm{i}\delta_-)} \tag{85}$$

where $\delta_\pm = (\boldsymbol{k}'\boldsymbol{k}' : \mathscr{D})^{1/2} \exp(\pm\mathrm{i}\theta')$.

In the broad spectrum approximation it is valid to write

$$\frac{\partial F^*_{k',\omega'}}{\partial \boldsymbol{p}} \frac{\partial F_{k',\omega'}}{\partial \boldsymbol{p}} = \frac{\boldsymbol{k}'\boldsymbol{k}'}{m^2} \left|\frac{\partial F_{k',\omega'}}{\partial \omega'}\right|^2. \tag{86}$$

From equation (85) we have

$$\left|\frac{\partial F_{k',\omega'}}{\partial \omega'}\right|^2 = \frac{\pi g(x)}{\boldsymbol{k}'\boldsymbol{k}' : \mathscr{D}} \tag{87}$$

where $x \equiv (\omega' - \boldsymbol{k}' \cdot \boldsymbol{v})/(\boldsymbol{k}'\boldsymbol{k}' : \mathscr{D})^{1/2}$, and

$$g(x) = \frac{(4/\pi)(x^2 + \cos^2\theta')}{|(x - \mathrm{i}\exp(\mathrm{i}\theta'))(x - \mathrm{i}\exp(-\mathrm{i}\theta'))|^4}.$$

Note that $g(x)$ falls off as $|x|^{-6}$ as $|x| \to \infty$. In the broad spectrum limit we can approximate $g(x)$ by a delta function

$$g(x) \simeq \rho_0(\theta')\,\delta(x) \tag{88}$$

where $\rho_0(\theta')$ is the integral of $g(x)$ from $-\infty$ to $+\infty$. Using contour integration we find $\rho_0(\theta') = (4 + s^2 + s^4)/4s$, where $s = \sec\theta'$.

Upon substitution of equations (86)–(88) in equation (80), it is found that

$$\mathscr{D} = \int\int \frac{\mathrm{d}^N k'\,\mathrm{d}\omega'}{(2\pi)^{N+1}} \frac{\pi\rho_0(\theta')\,\delta(\omega' - \boldsymbol{k}' \cdot \boldsymbol{v})}{(\boldsymbol{k}'\boldsymbol{k}' : \mathscr{D})^{1/2}} \frac{\boldsymbol{k}'\boldsymbol{k}'}{m^2} I_{k',\omega'}. \tag{89}$$

In the one-dimensional case the dispersion relation $\omega' = \omega_{k'}$ will pick out a single value of $|k'|$ for each value of v. Denoting this resonant wavevector by k_v we can solve

equation (89) for $\mathscr{D}$ to find

$$k_v^2\mathscr{D} = (\rho_0(\theta')k_v^2 D)^{2/3}, \tag{90}$$

where D is the quasilinear diffusion coefficient (Vedenov *et al* 1962, Drummond and Pines 1962):

$$D \equiv \int\int \frac{\mathrm{d}k'\,\mathrm{d}\omega'}{(2\pi)^2}\frac{(k')^2}{m^2}\pi\,\delta(\omega' - k'v)I_{k',\omega'}. \tag{91}$$

Substitution of equation (90) in equation (84) gives

$$\delta' = \cos\theta'(\rho_0(\theta')k_v^2 D)^{1/3}. \tag{92}$$

To within a constant of order unity equation (92) is in agreement with the width found by Dupree (1966), assuming purely velocity space diffusion. As our derivation was based on spatial diffusion it is clear that the two processes contribute approximately equally to resonance broadening in the region $k \sim k_v$, while we have shown that spatial diffusion is the dominant mechanism for $k \gg k_v$. Of course as $k \to 0$ it is clear from equation (74) that velocity diffusion becomes dominant. A method for obtaining an approximate solution for all values of k will be the subject of a subsequent paper.

7. Conclusions

We have developed a canonical perturbation theory for the single particle propagator which allows systematic calculation of non-linear effects and the inclusion of resonance broadening. The use of Green functions is reminiscent of quantum field theory and statistical mechanics (Kadanoff and Baym 1962); conversely, the Poisson bracket structure of the perturbation theory should allow straightforward quantization.

In this paper we have given the formal structure of the theory and indicated how it works in the simplest case: that of homogeneous, stationary electrostatic turbulence. There appears to be no reason why the theory cannot treat more complex situations, and in fact it is ideally suited for handling any problem which can be cast in Hamiltonian form. For instance the complex geometrical problems inherent in toroidal geometry can be simplified by using the theory of action-angle variables (Kaufman 1972). Another application is to the theory of coupled modes where the Hamiltonian now describes not a particle, but a set of coupled oscillators.

In order to apply the theory to plasma turbulence we must make the interaction Hamiltonian self-consistent with the particle distribution function through Maxwell's equations, and thus calculate linear and non-linear dielectric response functions, and hence wave–wave scattering coefficients (Johnston and Kaufman 1976). The theory will also be used to develop a kinetic theory along the lines of the dressed test particle method. This will require an extension of the Marcuvitz–Cook formalism to make the Green function self-consistent with Maxwell's equations, in order to include polarization effects.

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